Feedforward and Dynamic Uncoupling Control of Linear Multivariable Systems

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A modified V-form method is presented for the feedback dynamic uncoupling and feedforward control analysis of linear constant coefficient multivariable systems with the control objective of making two or more outputs invariant. Suitable feedback control degrees of freedom remain for a subsequent and independent primary feedback control analysis of each uncoupled subsystem. This method is also compared with the alternative structural analysis method of the authors.

Dynamic intercoupling or interaction between controlled process variables in multivariable systems prevents the direct application of single variable feedback control theory to such systems. If control could be introduced to eliminate intercoupling effects, the controlled multivariable process could be treated as several single variable subsystems. Moreover, if the dynamic uncoupling control left an uncommitted feedback control degree of freedom in each subsystem, a subsequent and independent primary feedback control analysis of each subsystem could be made by using well developed single variable techniques. Dynamic uncoupling control could, therefore, be useful to simplify the control analysis of multivariable processes.

Besides being a useful analytical tool, dynamic uncoupling has a beneficial influence on control quality. It prevents disturbances in one process variable from affecting others. Liu (*6*), in his study of dynamic uncoupling control of trajectory path processes, illustrates some of these effects. Under certain conditions, in fact, dynamic uncoupling control may be required for the optimal regulator control of linear multivariable processes.

It has been shown by the authors that dynamic intercoupling has both feedforward and feedback aspects. Accordingly, dynamic intercoupling controllers (even so-called "feedback" dynamic intercoupling controllers) serve both a feedforward and feedback role. These intercoupling controllers uncouple the variables only when their feedforward role is maximized (*4, 5*).

In his study of the optimal single variable composite feedforward-feedback control problem, Luecke (*7*) showed that, when process model error is small, the optimal policy is to use the maximum amount of feedforward control. Since the feedforward control role of dynamic uncoupling controllers is maximized, dynamic uncoupling control may well be the most suitable linear control method when model error is small.

The conventional transfer matrix representation (*P*-form) of a linear multivariable process expresses intercoupling between process variables in only a feedforward manner. By using this form, Bollinger and Lamb (*1*) developed a convenient and practical method for feedforward control analysis. However, they found that feedback uncoupling control could not be readily introduced into their analysis.

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Mesarovic (*8*) developed a system formulation (*V*-form) which expresses intercoupling as feedback in nature. The expression of intercoupling in a feedback form allows a convenient feedback uncoupling control analysis. However, the method cannot be directly applied to obtain both feedback uncoupling and feedforward controllers.

The objective of this paper is to combine these two approaches into a formulation which permits both a feedforward control analysis similar to the *P*-form and a feedback dynamic uncoupling control analysis similar to the *V*-form.

With the *V*-form of Mesarovic, there must be one independent manipulable process input for each dynamically uncoupled process variable. Input disturbance sources were not included so that the associated process transfer matrix is square. The definition of the *V*-form requires the inverse of this square process transfer matrix. However, the measurable disturbance inputs must be included in a formulation to provide for a feedforward control analysis. With an equal number of manipulable process inputs and controlled outputs, inclusion of the measurable disturbance inputs produces rectangular associated transfer matrices. Therefore, Mesarovic's definition of the *V*-form, requiring the inverse of the process transfer matrix, does not apply directly.

Foster and Stevens (*2, 3*) eliminated this difficulty with their *V'*-form by augmenting the rectangular process transfer matrix with arbitrary constants to provide a square transfer matrix. By classifying the process variables in the manner of Bollinger and Lamb (*1*), the present method (the modified *V*-form) will show that Mesarovic's *V*-form concept can be applied directly if only a submatrix of the process transfer matrix is square. The present method will also be briefly compared with the structural analysis method (*4, 5*) developed by the authors.

A MODIFIED V-FORM ANALYSIS

Assume a system which can be described by the set of linear differential equations

$$\frac{dy_i(t)}{dt} = \sum_{j=1}^{n} a_{ij}\, x_j(t) + \sum_{k=1}^{m} b_{ik}\, y_k(t) \quad i = 1, 2, \ldots, m \tag{1}$$

where the $y_i(t)$ are the state perturbation variables of the process (zero at the design steady state) and the $x_j(t)$ are the input perturbation variables of the process. It is assumed that the input variables are either manipulable or measurable. The Laplace transform of Equation (1) in matrix form is

$$sY_{m\text{X}1} = A_{m\text{X}n}X_{n\text{X}1} + B_{m\text{X}m}Y_{m\text{X}1} \tag{2}$$

Solving Equation (2) explicitly for $Y_{m\text{X}1}$

$$Y_{m\text{X}1} = P_{m\text{X}n}\, X_{n\text{X}1} \tag{3}$$

where

$$P_{m\text{X}n} = [s\, I_{m\text{X}m} - B_{m\text{X}m}]^{-1}\, A_{m\text{X}n} \tag{4}$$

With an equal number of manipulable inputs and controlled outputs, and assuming the vector $X_{n\text{X}1}$ contains some known disturbance inputs, n is greater than m. Therefore, $P_{m\text{X}n}$ is rectangular.

The process inputs and state variables can be classified according to the scheme of Bollinger and Lamb (*1*):

$X^{(K)}{}_{K\text{X}1}$ — the vector of K inputs which are measured or known, but which cannot be manipulated.

$X^{(M)}{}_{M\text{X}1}$ — the vector of M inputs which can be manipulated to achieve control.

$Y^{(N)}{}_{N\text{X}1}$ — the vector of N state variables which are neither measured (at least to actuate control) nor controlled.

$Y^{(C)}{}_{C\text{X}1}$ — the vector of C state variables which are to be dynamically uncoupled and controlled.

At least two additional classes of variables could be added which are not included in the present analysis:

$X^{(U)}{}_{U\text{X}1}$ — the vector of U inputs which are not measured or manipulated and therefore are unknown.

$Y^{(I)}{}_{I\text{X}1}$ — the vector of I internal state variables which are measured and used to actuate control, but are not themselves controlled.

The $X^{(U)}{}_{U\text{X}1}$ variables, which are of little significance in the present dynamic uncoupling and feedforward control analysis, would be important in any subsequent primary feedback control analysis. The $Y^{(I)}{}_{I\text{X}1}$ state variables, which were included in the structural analysis method of the authors (*5, 6*), are not considered here. Their inclusion is not essential to the concept of the modified *V*-form. However, both of these variable classes can be added to the method in a straightforward manner that parallels the present development.

If Equation (2) is properly ordered, it can be expressed in the partitioned form

$$s\begin{bmatrix} Y^{(N)}_{N\text{X}1} \\ \hdashline Y^{(C)}_{C\text{X}1}\end{bmatrix} = \begin{bmatrix} A^{(N,K)}_{N\text{X}K} & \vdots & A^{(N,M)}_{N\text{X}M} \\ \hdashline A^{(C,K)}_{C\text{X}K} & \vdots & A^{(C,M)}_{C\text{X}M}\end{bmatrix}\begin{bmatrix} X^{(K)}_{K\text{X}1} \\ \hdashline X^{(M)}_{M\text{X}1}\end{bmatrix} + \begin{bmatrix} B^{(N,N)}_{N\text{X}N} & \vdots & B^{(N,C)}_{N\text{X}C} \\ \hdashline B^{(C,N)}_{C\text{X}N} & \vdots & B^{(C,C)}_{C\text{X}C}\end{bmatrix}\begin{bmatrix} Y^{(N)}_{N\text{X}1} \\ \hdashline Y^{(C)}_{C\text{X}1}\end{bmatrix} \tag{5}$$

The uncontrolled state variables, $Y^{(N)}{}_{N\text{X}1}$, are not explicitly required in the control analysis and need be retained only implicitly in the formulation as

$$sY^{(C)}_{C\text{X}1} = [A(s)^{(C,K)}_{C\text{X}K} \vdots A(s)^{(C,M)}_{C\text{X}M}]\begin{bmatrix} X^{(K)}_{K\text{X}1} \\ \hdashline X^{(M)}_{M\text{X}1}\end{bmatrix} + B(s)^{(C,C)}_{C\text{X}C}\, Y^{(C)}_{C\text{X}1} \tag{6}$$

where

$$A(s)^{(C,K)}_{C\text{X}K} = A^{(C,K)}_{C\text{X}K} + G(s)^{(C,N)}_{C\text{X}N}\, A^{(N,K)}_{N\text{X}K} \tag{7}$$

$$A(s)^{(C,M)}_{C\text{X}M} = A^{(C,M)}_{C\text{X}M} + G(s)^{(C,N)}_{C\text{X}N}\, A^{(N,M)}_{N\text{X}M} \tag{8}$$

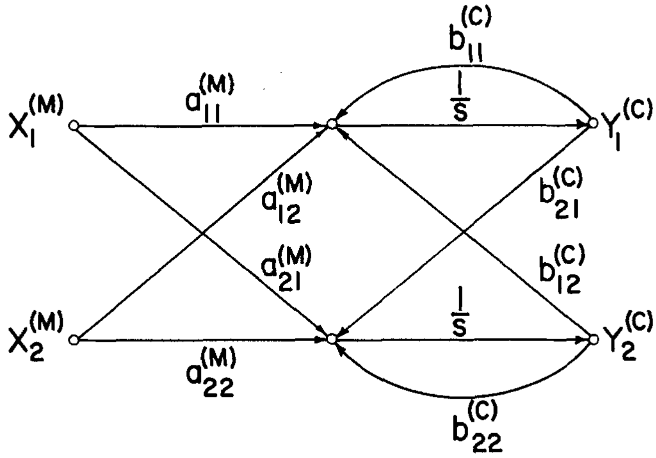

Fig. 1. Differential equation model.

$$B(s)^{(C,C)}_{C\text{X}C} = B^{(C,C)}_{C\text{X}C} + G(s)^{(C,N)}_{C\text{X}N}\, B^{(N,C)}_{N\text{X}C} \tag{9}$$

$$G(s)^{(C,N)}_{C\text{X}N} = B^{(C,N)}_{C\text{X}N}\, [s\, I_{N\text{X}N} - B^{(N,N)}_{N\text{X}N}]^{-1} \tag{10}$$

Equation (6) is of the same form as Equation (5) for the case where $N = 0$. Therefore, to simplify the notation, Equation (6) will be represented simply as

$$sY^{(C)}_{C\text{X}1} = [A^{(K)}_{C\text{X}K} \vdots A^{(M)}_{C\text{X}M}]\begin{bmatrix} X^{(K)}_{K\text{X}1} \\ \hdashline X^{(M)}_{M\text{X}1}\end{bmatrix} + B^{(C)}_{C\text{X}C}\, Y^{(C)}_{C\text{X}1} \tag{11}$$

where the coefficient matrices are functions of the Laplace variable when $N > 0$.

A simple case of Equation (11) ($K = N = 0$ and $C = M = 2$) is illustrated in Figure 1 where it is seen that the physical or differential equation model of the system contains both feedforward and feedback intercoupling mechanisms. The transfer matrix or *P*-form is found by solving explicitly for $Y^{(C)}_{C\text{X}1}$

$$Y^{(C)}_{C\text{X}1} = P^{(K)}_{C\text{X}K}\, X^{(K)}_{K\text{X}1} + P^{(M)}_{C\text{X}M}\, X^{(M)}_{M\text{X}1} \tag{12}$$

where the transfer matrices are defined by

$$[P^{(K)}_{C\text{X}K} \vdots P^{(M)}_{C\text{X}M}] = [s\, I_{C\text{X}C} - B^{(C)}_{C\text{X}C}]^{-1}\, [A^{(K)}_{C\text{X}K} \vdots A^{(M)}_{C\text{X}M}] \tag{13}$$

A simple case ($K = 0$ and $C = M = 2$) of the *P*-form is illustrated in Figure 2 where it can be observed that intercoupling is represented by only a feedforward mechanism.

The modified *V*-form that will be defined in this development is illustrated in Figure 3 for a simple case with known inputs and $C = M = 2$. Notice that all intercoupling is expressed by a feedback mechanism. This is the distinguishing characteristic of any *V*-form. The general

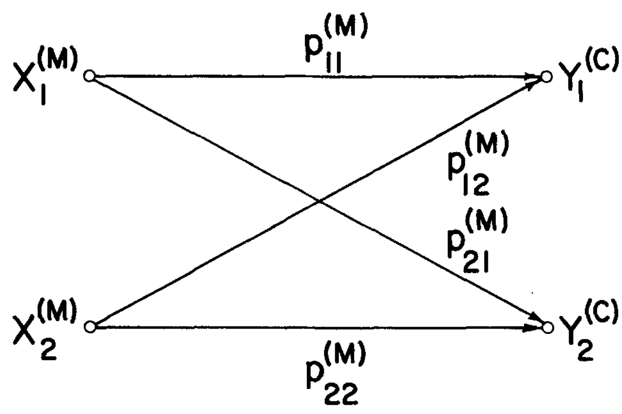

Fig. 2. P-form model.

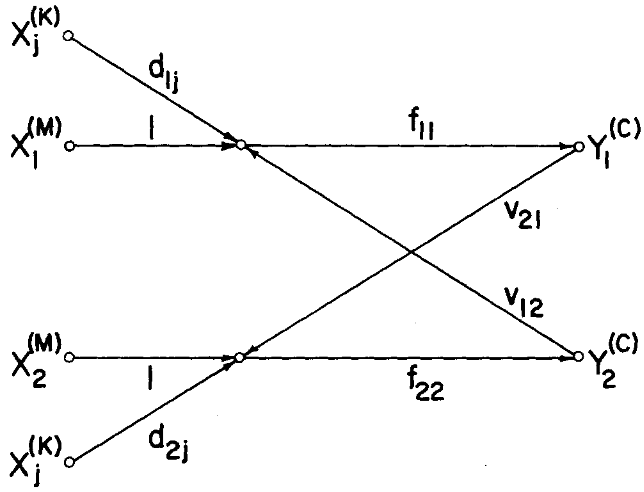

Fig. 3. Modified V-form model.

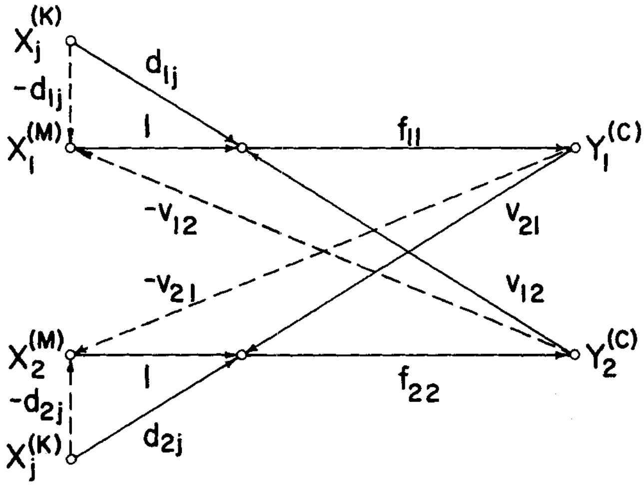

Fig. 4. Modified V-form with control.

matrix statement of this form is

$$Y^{(C)}_{CX1} = F^D_{CXC}[X^{(M)}_{MX1} + D_{CXK}X^{(K)}_{KX1} + V^\dagger_{CXC}Y^{(C)}_{CX1}] \qquad (14)^*$$

which can be related to the P-form by solving explicitly for $Y^{(C)}_{CX1}$.

$$Y^{(C)}_{CX1} = [I_{CXC} - F^D_{CXC}V^\dagger_{CXC}]^{-1}F^D_{CXC}[D_{CXK}X^{(K)}_{KX1} + X^{(M)}_{MX1}] \qquad (15)$$

Equating like coefficients of the inputs in Equations (12) and (15)

$$P^{(K)}_{CXK} = [I_{CXC} - F^D_{CXC}V^\dagger_{CXC}]^{-1}F^D_{CXC}D_{CXK} \qquad (16)$$

and

$$P^{(M)}_{CXM} = [I_{CXC} - F^D_{CXC}V^\dagger_{CXC}]^{-1}F^D_{CXC} \qquad (17)$$

Superficially, since there are three unknown matrices in Equation (16) and (17) and only two matrix equations, it might seem that the parameters of the modified V-form are indeterminant. However, by equating both sides of Equations (16) and (17) element by element, $C(C + K)$ equations are obtained and (with the off-diagonal of F^D_{CXC} and the diagonal elements of $V^{\dagger CXC}$ zero) there are really only $C(C + K)$ unknown elements. Therefore, the V-form is determinable.

The determination of the modified V-form parameters is quite straightforward. Premultiplying both sides of Equation (17) by $[I_{CXC} - F^D{}_{CXC}V^\dagger{}_{CXC}]$,

$$P^{(M)}_{CXM} - F^D_{CXC}V^\dagger_{CXC}P^{(M)}_{CXM} = F^D_{CXC} \qquad (18)$$

Assuming temporarily that $M = C$ and solving for $V^\dagger{}_{CXC}$,

$$V^\dagger_{CXC} = [F^D_{CXC}]^{-1} - [P^{(M)}_{CXM}]^{-1} \qquad (19)$$

The inverse of the diagonal matrix $F^D{}_{CXC}$ is also a diagonal matrix and the diagonal elements of $V^\dagger{}_{CXC}$ are zero. Therefore, Equation (19) states that

$$[F^D_{CXC}]^{-1} = \{[P^{(M)}_{CXM}]^{-1}\}^D \qquad (20)$$

so that

$$F^D_{CXC} = <\{[P^{(M)}_{CXM}]^{-1}\}^D>^{-1} \qquad (21)$$

and

$$V^\dagger_{CXC} = -\{[P^{(M)}_{CXM}]^{-1}\}^\dagger \qquad (22)$$

Using Equation (17) in Equation (16),

$$P^{(M)}_{CXM}D_{CXK} = P^{(K)}_{CXK} \qquad (23)$$

and since it has been assumed that $M = C$,

$$D_{CXK} = [P^{(M)}_{CXM}]^{-1}P^{(K)}_{CXK} \qquad (24)$$

Equations (21), (22), and (24) define the modified V-form parameters. Note that in this definition of the modified V-form only the submatrix $P^{(M)}_{CXM}$ of the total process transfer matrix is inverted and therefore needs to be both square and nonsingular. Since this submatrix is square if the process satisfies the present assumptions for the analysis (that is, one independent manipulable input for each process output to be controlled) the modified V-form is defined without arbitrary constants.

Introducing the control function

$$X^{(M)}_{MX1} = F^{(K)}_{MXK}X^{(K)}_{KX1} + F^{(C)}_{MXC}Y^{(C)}_{CX1} \qquad (25)$$

where

$$F^{(K)}_{MXK} = -D_{CXK} \qquad (26)$$

and

$$F^{(C)}_{MXC} = -V^\dagger_{CXC} \qquad (27)$$

into Equation (14) provides both feedforward and dynamic uncoupling control,

$$Y^{(C)}_{CX1} = F^D_{CXC}[-D_{CXK}X^{(K)}_{KX1} - V^\dagger_{CXC}Y^{(C)}_{CX1} + D_{CXK}X^{(K)}_{KX1} + V^\dagger_{CXC}Y^{(C)}_{CX1}] \qquad (28)$$

or

$$Y^{(C)}_{CX1} = F^D_{CXC}[O_{CXK}X^{(K)}_{KX1} + O_{CXC}Y^{(C)}_{CX1}] \qquad (29)$$

Of course, in any actual system there would be errors due to linearization and approximation of the system dynamics so that Equation (29) will only be approximately true. Accordingly, it will be necessary to introduce additional feedback control. This is possible since C feedback control degrees of freedom (the diagonal elements of $F^{(C)}_{MXC}$) have been left uncommitted. Moreover, the design of these additional feedback controllers can be conducted independently since the system has already been dynamically

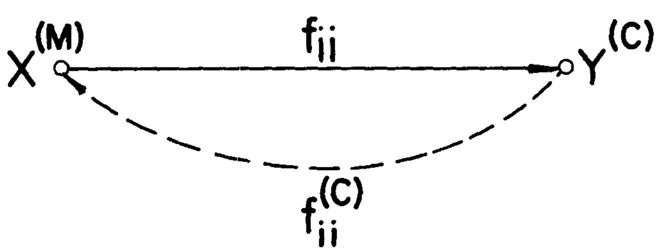

Fig. 5. Feedback control of uncoupled subsystem.

* The D superscript denotes either a matrix defined as a diagonal matrix or a diagonal matrix formed of the diagonal elements of the unsuperscripted matrix. Similarly, a dagger superscript indicates a matrix with a zero diagonal, either by definition or as composed of the off-diagonal elements of the unsuperscripted matrix.

uncoupled. The relationship of the feedforward and dynamic uncoupling controllers to the process system is illustrated in Figure 4 where the control loops are shown as dashed lines. Feedback stabilization of the C independent, noninteracting subsystems is illustrated in Figure 5.

GENERALIZATION OF THE MODIFIED V-FORM

The modified V-form as now defined would have some useful applications. However, the V-form concept can be generalized somewhat so as to increase its usefulness. Up to this point, a modified V-form control analysis produces a controlled system consisting of C noninteracting subsystems. Each one of these subsystems contains one manipulable input and one controlled output so that a primary feedback control analysis of each subsystem can be conducted separately. Each of the manipulable inputs to the subsystems represents an actual physical variable and has a limited (due to constraints) control effort capability associated with it which is available for primary feedback control. The control effort requirements for the primary feedback control will differ from subsystem to subsystem. Thus, it is entirely conceivable that some subsystems would have an insufficient amount of available control effort while others would have a surplus. The concept of the V-form does not require that the manipulable inputs to the subsystem be single physical variables. The subsystem manipulable inputs could as well be linear combinations of the actual, physical manipulable inputs. This refinement of the method is accomplished quite easily by introducing

$$X^{(M)}_{MX1} = W_{MXC}\, X'^{(M)}_{CX1} \tag{30}$$

where $X'^{(M)}_{MX1}$ is a vector of combined manipulable inputs (the subsystem manipulable inputs) and W_{MXC} is a weighting matrix. The weighting matrix can be defined simply as that matrix which allocates control effort demands between the actual manipulable inputs to allow a degree of conformance with the problem constraints.

By defining combined manipulable inputs as in Equation (30), a modified V-form is obtained which is similar to the less general case. By using Equation (30) in Equation (12) we get

$$Y^{(C)}_{CX1} = P^{(K)}_{CXK}\, X^{(K)}_{KX1} + P^{(M)}_{CXM}\, W_{MXC}\, X'^{(M)}_{CX1} \tag{31}$$

so that $P^{(M)}_{CXM}\, W_{MXC}$ serves the role previously associated with $P^{(M)}_{MXC}$ alone. Therefore, a more general modified V-form is defined by Equations (32) through (34).

$$F^{D}_{CXC} = <\{[P^{(M)}_{CXM}\, W_{MXC}\,]^{-1}\}^{D}>^{-1} \tag{32}$$

$$V^{\dagger}_{CXC} = -\{[P^{(M)}_{CXM}\, W_{MXC}\,]^{-1}\}^{\dagger} \tag{33}$$

$$D_{CXK} = [P^{(M)}_{CXM}\, W_{MXC}\,]^{-1}\, P^{(K)}_{CXK} \tag{34}$$

The introduction of W_{MXC} also produces $P^{(M)}_{MXC}\, W_{MXC}$ matrices which are always square. Therefore, Equations (32) through (34), which define the modified V-form, also apply for the cases where $M > C$. The $P^{(M)}_{CXM}\, W_{MXC}$ matrix is also square when $M < C$, but it does not possess an inverse because its rank is less than its order. Thus, the modified V-form (or any V-form) does not exist when $M < C$.

A detailed discussion of factors which might influence the selection of W_{MXC} is beyond the scope of this paper. However, this matrix has three distinct roles. One of these is the specification of the relative weighting of the physical manipulable inputs in the combined manipulable inputs. The simplest example of this role is

$$W_{MXC} = I_{CXC} \tag{35}$$

where the combined manipulable input vector is identical to the physical manipulable input vector. Another role is manipulable input selection for cases where $M > C$. The simplest case of this is

$$W_{MXC} = \begin{bmatrix} I_{CXC} \\ \hdashline O_{(M-C)XC} \end{bmatrix} \tag{36}$$

Here the combined manipulable input vector consists of the first C physical manipulable inputs. The third role is that of system ordering with an example being

$$W_{MXC} = \begin{pmatrix} O & & \\ & 1 & \\ & & O \end{pmatrix} \tag{37}$$

In this case, the manipulable input arbitrarily designated as the first manipulable input, $X^{(M)}_{1}$, will be associated with the Cth subsystem rather than the first subsystem, and so forth.

Equations (32) through (34) define a more general modified V-form and provide an organized framework for the solution of the problem. Organization is extremely important because problem complexity increases rapidly with dimensionality. The controller design equations [Equations (32) to (34) and (26) to (27)] can be directly evaluated at discrete frequencies in the complex frequency domain by using a digital computer (*1*). Analytic functions may then be fitted to the resulting numerical data (9). The evaluation of various modified V-form configurations can be carried out simply by using alternative W_{MXC} matrices, and often with little additional calculation. For example, when $M = C$,

$$[P^{(M)}_{CXM}\, W_{MXC}\,]^{-1} = [W_{MXC}\,]^{-1}\, [P_{CXM}\,]^{-1} \tag{38}$$

so that the determination of feedforward and dynamic uncoupling controllers for an additional configuration requires only the inversion of an additional constant matrix (when W_{MXC} is constant). This is significant because numerical matrix inversions are relatively time consuming.

EXAMPLE: MODIFIED V-FORM ANALYSIS

The above analysis will now be applied to the example problem of Foster and Stevens (3). The chemical reactor of this example can be represented by the linearized, Laplace transformed, matrix equation

$$s\begin{Bmatrix} Y^{(C)}_{1} \\ Y^{(C)}_{2} \end{Bmatrix} = \begin{bmatrix} 0 & a^{(K)}_{12} \\ a^{(K)}_{21} & 0 \end{bmatrix}\begin{Bmatrix} X^{(K)}_{1} \\ X^{(K)}_{2} \end{Bmatrix} + \begin{bmatrix} a^{(M)}_{11} & 0 \\ a^{(M)}_{21} & a^{(M)}_{22} \end{bmatrix}\begin{Bmatrix} X^{(M)}_{1} \\ X^{(M)}_{2} \end{Bmatrix} + \begin{bmatrix} b^{(C)}_{11} & b^{(C)}_{12} \\ b^{(C)}_{21} & b^{(C)}_{22} \end{bmatrix}\begin{Bmatrix} Y^{(C)}_{1} \\ Y^{(C)}_{2} \end{Bmatrix} \tag{39}$$

where $N = 0$ and $M = C = 2$. It is assumed that there are no critical manipulable input constraints. From Equation (13) we obtain

$$[P^{(M)}_{2x2}\,]^{-1} = [A^{(M)}_{2x2}\,]^{-1}\, [s\, I_{2x2} - B^{(C)}_{2x2}\,] \tag{40}$$

or

$$[P^{(M)}_{2x2}\,]^{-1} = \begin{bmatrix} \dfrac{s - b^{(C)}_{11}}{a^{(M)}_{11}} \\ -\left[\dfrac{a^{(M)}_{21}\,[s - b^{(C)}_{11}]}{a^{(M)}_{11}a^{(M)}_{22}} + \dfrac{b^{(C)}_{21}}{a^{(M)}_{22}}\right] \end{bmatrix}$$

$$\left.\begin{matrix} -\dfrac{b_{12}^{(C)}}{a_{11}^{(M)}} \\ \left[\dfrac{a_{21}^{(M)}\, b_{12}^{(C)}}{a_{11}^{(M)}\, a_{22}^{(M)}} + \dfrac{[s-b_{22}^{(C)}]}{a_{22}^{(M)}}\right] \end{matrix}\right] \tag{41}$$

Using this result with Equations (22) and (27)

$$V^{\dagger}_{2x2} = -F^{(C)}_{2x2} = \begin{bmatrix} 0 & \dfrac{b_{12}^{(C)}}{a_{11}^{(M)}} \\ \dfrac{a_{21}^{(M)}[s-b_{11}^{(C)}] + a_{11}^{(M)}\, b_{21}^{(C)}}{a_{11}^{(M)}\, a_{22}^{(M)}} & 0 \end{bmatrix} \tag{42}$$

and by Equation (21)

$$F^{D}_{2x2} = \begin{bmatrix} \dfrac{a_{11}^{(M)}}{s-b_{11}^{(C)}} & 0 \\ 0 & \dfrac{a_{11}^{(M)}\, a_{22}^{(M)}}{a_{11}^{(M)}[s-b_{22}^{(C)}] + a_{21}^{(M)}\, b_{12}^{(C)}} \end{bmatrix} \tag{43}$$

From Equations (24) and (13)

$$D_{CXK} = [A^{(M)}_{CXM}]^{-1} A^{(K)}_{CXK} \tag{44}$$

and with Equations (39) and (26)

$$D_{CXK} = -F^{(K)}_{MXK} = \begin{bmatrix} 0 & \dfrac{a_{12}^{(K)}}{a_{11}^{(M)}} \\ \dfrac{a_{21}^{(K)}}{a_{22}^{(M)}} & -\dfrac{a_{21}^{(M)}\, a_{12}^{(K)}}{a_{11}^{(M)}\, a_{22}^{(M)}} \end{bmatrix} \tag{45}$$

COMPARISON WITH THE V′-FORM

In order to distinguish the present method from the V'-form of Foster and Stevens, their approach will be briefly reviewed here. As discussed in the introduction, the central problem in applying Mesarovic's V-form concept is the formulation of the problem such that the transfer matrix to be inverted in the V-form definition is square and nonsingular. Instead of partitioning this matrix, the V'-form analysis augments the process transfer matrix [Equation (3)] with arbitrary constants so that it becomes square. That is,

$$\begin{bmatrix} Y_{mX1} \\ \hdashline Y^{(V)}_{(n-m)X1} \end{bmatrix} = \begin{bmatrix} P_{MXn} \\ \hdashline K_{(n-m)X1} \end{bmatrix} X_{nX1} \tag{46}$$

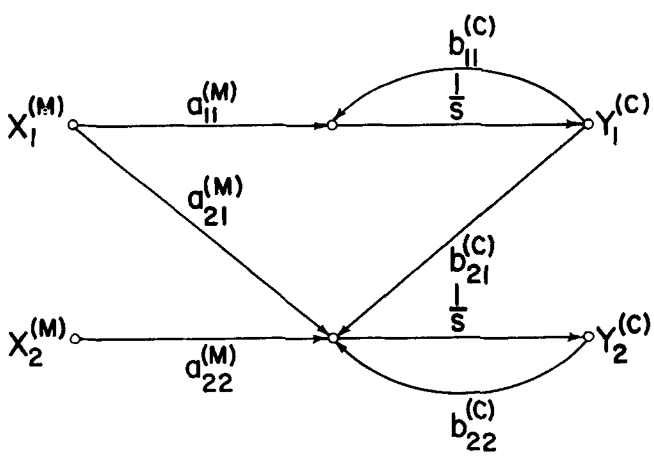

Fig. 6. A differential equation model.

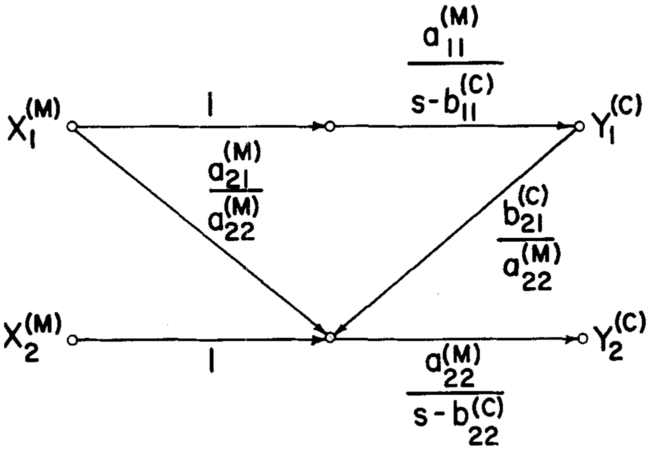

Fig. 7. Simplified differential equation model.

where the submatrix $K_{(n-m)Xn}$ consists of arbitrary constants, and the subvector $Y^{(V)}_{(n-m)X1}$ contains virtual outputs which are related to the input matrix by the arbitrary constants. The final design relationships still contain these arbitrary constants.

The V'-form analysis concludes:

1. The uncoupled subsystem transfer functions are always first-order.
2. The feedback uncoupling controllers are simple proportional-derivative controllers.
3. The feedforward controllers are simple proportional controllers.

These conclusions require the assumption that all the state variables are either measured and used to actuate control or both controlled and measured to actuate control ($N = 0$).

The V'-form controllers with the arbitrary constants would be of higher order than the modified V-form controllers when $N > 0$. Only a unique set of these arbitrary constants would reduce the order to that of the modified V-form analysis.

As mentioned in the introduction, one of the incentives for the use of a V-type analysis to achieve dynamic uncoupling is to allow a subsequent, independent primary feedback control analysis of each subsystem. The arbitrary constants of the V'-form do not permit such independent, serial analysis.

LIMITATIONS OF ANY V-TYPE ANALYSIS

If there are any control element time lags requiring compensation, additional derivative action must be included in the controllers (*1*). The feedback uncoupling controllers would then contain second derivative control modes which would usually be unrealizable. In addition, first derivative control modes can be a problem in the presence of moderate measurement noise and model error (*7*). These problems are further compounded when cases are considered where $N > 0$.

The origin of the derivative modes in the feedback intercoupling loops is apparent from a comparison of Figures 1 and 3. Figure 1 is an illustration of the differential equation representation of the system and contains both feedforward and feedback intercoupling mechanisms. Figure 3, which illustrates the V-form, contains only feedback intercoupling and is, therefore, somewhat less realistic. The derivative mode occurs because the origins of the feedforward intercoupling loops are translated through an integrator loop. To compensate for this, the resulting feedback intercoupling loops must contain a derivative mode. This translation process is illustrated in Figures 6 through 9. Figure 6 is the same system as Figure 1 but where $a_{12}^{(M)} = b_{12}^{(C)} = 0$ for clarity. Figure 7 is a simplification of the system before the translation. In Figure 8, the origin of the feedforward intercoupling loop has been translated to $Y_1^{(C)}$. In Figure 9, the translated feedforward intercoupling loop and the feedback intercoupling loop are combined. The resulting configuration gives $V^{\dagger}_{2x2}$ and F^{D}_{2x2} matrices which correspond with Equations (42) and (43) when $b_{12}^{(C)}$ is set equal to zero.

The preceding discussion tends to indicate that the *V*-form and the differential equations are not equivalent models. In fact they actually describe two different physical systems since the inverse Laplace transform of the *V*-form (and also the *P*-form) does not reduce to the system differential equations. The *V*-form does have the same input-output (terminal) relationship as the differential equation model. But, there are an infinite number of forms which have the same terminal relationship (*8*).

The basic difference between the *P* and *V*-forms and the differential equation model is that the latter contains more process information. *P* and *V*-forms can be obtained from the differential equation model, but the differential equation model cannot be directly obtained from the *P* and *V*-forms. The additional information of the differential equation model is apparent in its more detailed description of system interactions. Since the derivative mode dynamic uncoupling controllers are a consequence of the *V*-form description of interaction, the limitations of the *V*-form are ultimately due to its limited information content. However, the *V*-form controllers are the best that can be obtained analytically if only plant input-output information is used. Therefore, if the limitations of the *V*-form are to be avoided, more information and a formulation which effectively expresses this information is needed.

RELATION BETWEEN V-TYPE AND STRUCTURAL ANALYSES

An alternative to the *V*-type analyses uses a structural system formulation which contains more information (*4*, *5*). This method considers intercoupling which appears as feedforward intercoupling in the differential equation model as feedforward intercoupling, and intercoupling which appears as feedback intercoupling as feedback intercoupling. Such an approach avoids the derivative mode dynamic uncoupling controllers and is more generally applicable. For example, it permits the minimization of intercoupling when $M < C$.

The structural formulation is essentially the same as the

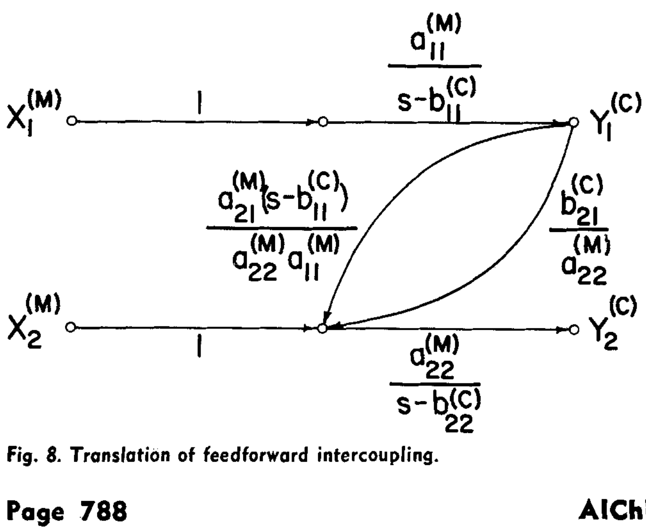

Fig. 8. Translation of feedforward intercoupling.

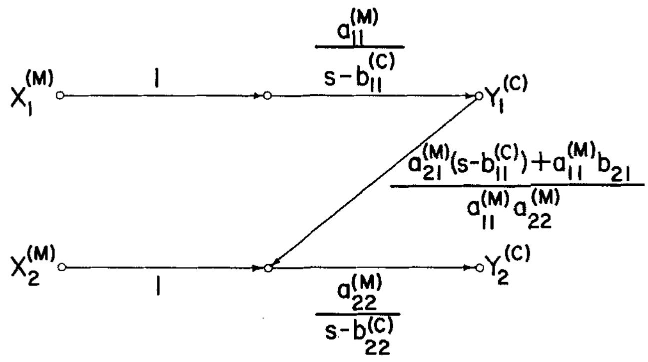

Fig. 9. Combined intercoupling gives V-form.

differential equation model for simple cases such as the example problem of this paper where $N = 0$. However, in more general cases the structural formulation contains less information than the differential equation model but includes all the information that can be used in the control analysis.

In the chemical reactor model of Equation (39), the feedback intercoupling mechanism can be separated from direct feedback as follows

$$\begin{bmatrix} Y_1^{(C)} \\ Y_2^{(C)} \end{bmatrix} = \begin{bmatrix} \frac{1}{s-b_{11}^{(C)}} & 0 \\ 0 & \frac{1}{s-b_{22}^{(C)}} \end{bmatrix} \left\{ \begin{bmatrix} 0 & a_{12}^{(K)} \\ a_{21}^{(K)} & 0 \end{bmatrix} \begin{bmatrix} X_1^{(K)} \\ X_2^{(K)} \end{bmatrix} + \begin{bmatrix} a_{11}^{(M)} & 0 \\ a_{21}^{(M)} & a_{22}^{(M)} \end{bmatrix} \begin{bmatrix} X_1^{(M)} \\ X_2^{(M)} \end{bmatrix} + \begin{bmatrix} 0 & b_{12}^{(C)} \\ b_{21}^{(C)} & 0 \end{bmatrix} \begin{bmatrix} Y_1^{(C)} \\ Y_2^{(C)} \end{bmatrix} \right\} \tag{47}$$

This expression can be normalized with respect to the diagonal elements of the $A_{2x2}^{(M)}$ matrix,

$$\begin{bmatrix} Y_1^{(C)} \\ Y_2^{(C)} \end{bmatrix} = \begin{bmatrix} \frac{a_{11}^{(M)}}{s-b_{11}^{(C)}} & 0 \\ 0 & \frac{a_{22}^{(M)}}{s-b_{22}^{(C)}} \end{bmatrix} \left\{ \begin{bmatrix} 0 & \frac{a_{12}^{(K)}}{a_{11}^{(M)}} \\ \frac{a_{21}^{(K)}}{a_{22}^{(M)}} & 0 \end{bmatrix} \begin{bmatrix} X_1^{(K)} \\ X_2^{(K)} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \frac{a_{21}^{(M)}}{a_{22}^{(M)}} & 1 \end{bmatrix} \begin{bmatrix} X_1^{(M)} \\ X_2^{(M)} \end{bmatrix} + \begin{bmatrix} 0 & \frac{b_{12}^{(C)}}{a_{11}^{(M)}} \\ \frac{b_{21}^{(C)}}{a_{22}^{(M)}} & 0 \end{bmatrix} \begin{bmatrix} Y_1^{(C)} \\ Y_2^{(C)} \end{bmatrix} \right\} \tag{48}$$

Equation (48) is the structural formulation for this simple case, and is illustrated by the solid lines of Figure 10. In Figure 10, it can be observed that feedforward intercoupling has been maintained as feedforward intercoupling and feedback intercoupling has been maintained as feedback intercoupling. The difference in the control analysis between this method and the modified *V*-form method is that feedforward intercoupling can be eliminated by feedforward uncoupling controllers. The resulting control configuration is illustrated by the dashed lines in Figure 10. The feedforward uncoupling controller, $f_{21}^{(M)}$, is

$$f_{21}^{(M)} = -\frac{a_{21}^{(M)}}{a_{22}^{(M)}} \tag{49}$$

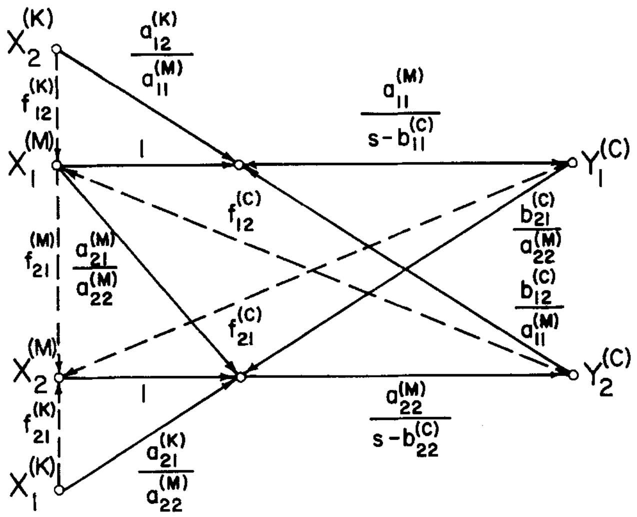

Fig. 10. Structural model with control.

the feedback uncoupling controllers are

$$f_{12}^{(C)} = -\frac{b_{12}^{(C)}}{a_{11}^{(M)}} \; ; \; f_{21}^{(C)} = -\frac{b_{21}^{(C)}}{a_{22}^{(M)}} \tag{50}$$

and the feedforward compensation controllers are

$$f_{12}^{(K)} = -\frac{a_{12}^{(K)}}{a_{11}^{(M)}} \; ; \; f_{21}^{(K)} = -\frac{a_{21}^{(K)}}{a_{22}^{(M)}} \tag{51}$$

An important point to notice is that all the controllers in this simple example are proportional because $N = 0$.

A simple relationship can be established between the above controllers and those obtained by the modified V-form analysis. The role of the feedforward uncoupling controller $f_{21}^{(M)}$ is to produce a new system that is the same in every respect except that there is no feedforward intercoupling. That is, in Equation (39) the feedforward intercoupling parameter $a_{21}^{(M)}$ becomes effectively zero. Letting $a_{21}^{(M)}$ equal zero in Equations (42), (43), and (45), the modified V-form feedback uncoupling and feedforward compensation controllers would correspond exactly with the structural controllers of Equations (50) and (51).

CONCLUSIONS

The modified V-form control analysis provides feedforward and dynamic uncoupling control of linear multivariable processes in a direct, practical manner. Suitable feedback control degrees of freedom remain for a subsequent primary feedback control analysis of each uncoupled subsystem using well developed single variable techniques.

While the structural analysis method provides a better control configuration than the modified V-form, the modified V-form may use a process model of lower informational content. In those cases where only plant transfer information is available and where the limitations of the V-form are not critical, it may be more convenient to use the modified V-form analysis than to determine the additional information needed for a structural analysis. Therefore, the limited information content of the modified V-form is the source of both its advantages and disadvantages as compared to the structural analysis.

NOTATION

a_{ij} = an input coefficient
b_{ik} = a state variable coefficient
C = the number of controlled state variables
I = the number of control actuating internal process variables
K = the number of known inputs
m = the number of state variables of a process
M = the number of manipulable inputs
n = the number of inputs to a process
N = the number of state variables which are neither controlled nor actuate control
s = the Laplace variable
t = time
U = the number of unmeasured and unmanipulated process inputs
x_j = an input to a process
y_k = a state variable of a process

Matrices

A = an input coefficient matrix
B = a state variable coefficient matrix
D = the coefficient matrix of the known inputs in the modified V-form
F = a control matrix
F^D_{CXC} = the subsystem transfer matrix in the modified V-form
I = the identity matrix
P = a plant transfer matrix
V = the feedback intercoupling matrix in the modified V-form
W = an input weighting matrix
X = an input vector
X' = a combined input vector
Y = a state variable vector
O = a null matrix

Superscripts and Subscripts

C = denotes the number and identity of controlled state variables
D = designates a diagonal matrix composed of the principal diagonal of the square argument matrix
I = denotes the number and identity of control actuating internal process variables
K = denotes the number and identity of known inputs
m = the number of state variables of a process
M = denotes the number and identity of manipulable inputs
n = the number of inputs to a process
N = denotes the number and identity of state variables which are neither controlled nor actuate control
U = denotes the number and identity of unmeasured, unmanipulable process inputs
$\dagger$ = designates the square argument matrix with the principal diagonal set equal to zero

LITERATURE CITED

1. Bollinger, R. E., and D. E. Lamb, *Chem. Eng. Progr. Symp. Ser. No. 55*, **61**, 66 (1965).
2. Foster, R. D., and W. F. Stevens, *AIChE J.*, **13**, 334 (1967).
3. *Ibid.*, **13**, 340 (1967).
4. Greenfield, G. G., and T. J. Ward, *Ind. Eng. Chem. Fundamentals*, **6**, 571 (1967).
5. *Ibid.*, **6**, 564 (1967).
6. Liu, S. L., *Ind. Eng. Chem. Process Design Develop.*, **6**, 460 (1967).
7. Luecke, R. H., Ph.D. dissertation, Univer. Oklahoma, Norman (1966).
8. Mesarovic, M. D., *"The Control of Multivariable Systems,"* Mass. Inst. Tech. Press, Cambridge, Mass., (1960).
9. Simpkins, C. R., Ph.D. dissertation, Univer. Delaware, Newark (1966).

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